General Topology and its Applications 6 (1976) 319–326

A NOTE ON THE G_δ-CLOSURE AND THE REALCOMPACTNESS OF 2^X

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Received 16 December 1974

In this note we describe the G_δ-closure of 2^X in $2^{\beta X}$. Using this description, we obtain the following theorem: 2^X is G_δ-closed in $2^{\beta X}$ if and only if X is Lindelöf. An immediate consequence of this result is the fact that 2^X is realcompact if X is Lindelöf. Concerning the realcompactness of 2^X, we show that the class of completely regular spaces X, for which 2^X is realcompact, is closed under continuous-open-closed images. In particular, if X is completely regular and 2^X is realcompact, then any continuous-open-closed completely regular image of X is realcompact.

AMS Subj. Class.: Primary 54B20; Secondary 54C50, 54D30, 54D60

hyperspace	G_δ-closure
Lindelöf space	realcompact space

1. Introduction

Let X be a completely regular space,[1] and let 2^X denote the space of closed subsets of X with the finite topology (see [4] and [6] for the fundamental properties of 2^X). Let $i : 2^X \to 2^{\beta X}$ be the canonical mapping defined by $i(F) = \mathrm{cl}_{\beta X} F$. Then i is a continuous map from 2^X onto a dense subspace of $2^{\beta X}$, and, as is demonstrated in [3], i is an embedding if and only if X is normal. Thus, if X is normal, $2^{\beta X}$ is a compactification of 2^X. In this note we describe the G_δ-closure of 2^X in $2^{\beta X}$, and establish the following theorem: 2^X is G_δ-closed in $2^{\beta X}$ if and only if X is Lindelöf. A corollary of this theorem is that 2^X is realcompact if X is Lindelöf. Concerning the realcompactness of 2^X, we show that the class of com-

* This research was partially supported by a fellowship from the National Research Council of Canada.

[1] In this paper, the term completely regular, even when unmodified, implies Hausdorff.

pletely regular spaces X for which 2^X is realcompact is closed under continuous-open-closed images. In particular, if 2^X is realcompact, then every continuous-open-closed image of X is realcompact.

For the reader's convenience, we recall the definition of 2^X. For a topological space X, 2^X denotes the set of all non-empty closed subsets of X. For a subset A of X we denote the set $\{F \in 2^X : F \subseteq A\}$ by 2^A. We generate a topology on 2^X by taking the family

$$\{2^G : G \text{ is open in } X\} \cup \{2^X - 2^F : F \text{ is closed in } X\}$$

as a sub-basis. This topology on 2^X is known as the finite topology, and 2^X with this topology is referred to as the hyperspace of X. Following [4], we adapt the following notational convention. For subsets $A_0, A_1, ..., A_n$ of X we let

$$B(A_0; A_1, A_2, ..., A_n) = 2^{A_0} \cap \bigcap_{i=1}^{n} 2^X - 2^{X-A_i}$$

$$= \{F : F \subseteq A_0 \text{ and } F \cap A_i \neq \emptyset \text{ for all } i = 1, 2, ..., n\} .$$

Using this notation, we see that the family of sets of the form $B(G_0; G_1, G_2, ..., G_n)$, where $G_0, G_1, ..., G_n$ are open in X and $\bigcup_{i=1}^{n} G_i \subseteq G_0$, forms a basis for the topology on 2^X.

2. The G_δ-closure of 2^X in $2^{\beta X}$

Let X be a subspace of a space Y. We say that X *is* G_δ*-closed in* Y if $Y - X$ is a union of G_δ-sets in Y. This means that given a point $p \in Y - X$ we can find open subsets $G_1, G_2, ...$ of Y such that $p \in \bigcap_{n \in \mathbb{N}} G_n \subseteq Y - X$. The following theorem is well known (see [1, 8.8]).

Theorem 2.1. *Let X be a completely regular Hausdorff space. Then the following statements are equivalent*:

(i) *X is realcompact,*

(ii) *X is G_δ closed in βX,*

(iii) *X is G_δ-closed in some compactification of X*

Of course a realcompact space need not be G_δ-closed in all of its compactifications. For example, a locally compact space is G_δ-closed in its one-point compactification if and only if it is σ-compact, and examples abound of realcompact, locally compact spaces which are not σ-compact.

Suppose we are given a compactification αX of a (completely regular Hausdorff) space X. If X is G_δ-closed in αX, we see, by 2.1, that X is realcompact. As we saw above, the G_δ-closedness of X in αX may reflect much more than the realcompactness of X, depending on the position of αX in the family of all compactifications of X.

Now, let X be a normal, Hausdorff space. As was mentioned above, the mapping $i(F) = \mathrm{cl}_{\beta X} F$ is an embedding of 2^X onto a dense subspace of $2^{\beta X}$, and so, in this way, we can regard $2^{\beta X}$ as a compactification of 2^X. We are concerned here with determining the topological structure of X that is equivalent to 2^X being G_δ-closed in $2^{\beta X}$. We begin with an elementary observation.

Proposition 2.2. *Let X be a subspace of Y. t*

$$Q_Y(X) = \{p \in Y\colon \text{every } G_\delta\text{-set in } \quad \text{containing } p \text{ intersects } X\}\,.$$

Then $Q_Y(X)$ is the smallest subspace of Y th t contains X and is G_δ-closed in Y.

Proof. Clearly $X \subseteq Q_Y(X)$. We first show that $Q_Y(X)$ is G_δ-closed in Y. For, let $p \in Y - Q_Y(X)$. Then there is a G_δ-set H in Y such that $p \in H$ and $H \cap X = \emptyset$. Since any G_δ containing any point of $Q_Y(X)$ intersects X, we have $H \cap Q_Y(X) = \emptyset$. Thus $Q_Y(X)$ is G_δ-closed in Y.

Now, let S be any subspace of Y such that $X \subseteq S$ and S is G_δ-closed in Y. We show that $Q_Y(X) \subseteq S$. Let $p \in Q_Y(X)$. If $p \notin S$, since S is G_δ-closed in Y, there is a G_δ-set H in Y such that $p \in H$ and $H \cap S = \emptyset$. But since $p \in Q_Y(X)$, $H \cap X \neq \emptyset$, and since $X \subseteq S$, $H \cap S \neq \emptyset$. Thus $p \in S$. Therefore $Q_Y(X) \subseteq S$, and so $Q_Y(X)$ is the smallest G_δ-closed subspace of Y which contains X. □

We will refer to $Q_Y(X)$ in 2.2 as the G_δ-*closure of X in Y*.

For a topological space X, let $C(X)$ denote the set of continuous, real-valued functions on X. For $f \in C(X)$ we let

$$Z(f) = \{x \in X\colon f(x) = 0\}.$$

$Z(f)$ is called the zero-set of f. We let

$$Z(X) = \{Z(f)\colon f \in C(X)\}\,.$$

In a completely regular space, a G_δ-set containing a given point contains a zero-set containing the given point (see [1, 3.11]). So if $X \subseteq Y$, where Y is completely regular, our G_δ conditions may be re-formulated as follows. X is G_δ-closed in Y if every point of $Y - X$ lies in some zero-set of Y that is disjoint from X. $Q_Y(X)$ is the set of points p such that every zero-set in Y that contains p intersects X.

We now describe the G_δ-closure of 2^X in $2^{\beta X}$ for a normal space X. Recall that we are identifying 2^X as the subspace $i(2^X)$ of $2^{\beta X}$.

Lemma 2.3. *Let X be a normal, Hausdorff space. Let Q denote the G_δ-closure of 2^X in $2^{\beta X}$. Then*

$$Q = \{F \in 2^{\beta X} : Z \in Z(\beta X), F \subseteq Z \Rightarrow F \subseteq \mathrm{cl}_{\beta X}(Z \cap X)\}\ .$$

Proof. Observe that $i(2^X) = \{F \in 2^{\beta X} \colon F = \mathrm{cl}_{\beta X}(F \cap X)\}$. Let Q_1 denote the set described in the statement of the lemma. We will show that $2^X \subseteq Q_1 \subseteq Q$ and that Q_1 is G_δ-closed in $2^{\beta X}$, from which the assertion follows. Since $F \in 2^X$ is equivalent to $F = \mathrm{cl}_{\beta X}(F \cap X)$, clearly $2^X \subseteq Q_1$. We now show that $Q_1 \subseteq Q$. So let $F \in Q_1$. Let $\mathcal{H}$ be any G_δ-set in $2^{\beta X}$ containing F. Write $\mathcal{H} = \bigcap_{n\in\mathrm{N}} \mathcal{G}_n$, with $\mathcal{G}_n$ open in $2^{\beta X}$ for each n. For each n we can find open sets $G_{0,n}, G_{1,n}, ..., G_{K_n,n}$ in βX with $\bigcup_{i=1}^{K_n} G_{i,n} \subseteq G_{0,n}$ and $F \in B(G_{0,n}; G_{1,n}, ..., G_{K_n,n}) \subseteq \mathcal{G}_n$. For each n find a zero-set Z_n in βX such that $F \subseteq Z_n \subseteq G_{0,n}$. Now $\bigcap_{n\in\mathrm{N}} Z_n = Z$ is a zero-set in βX and $F \subseteq Z$. Since $F \in Q_1$, we have $F \subseteq \mathrm{cl}_{\beta X}(Z \cap X)$. But $\mathrm{cl}_{\beta X}(Z \cap X) \in 2^X \cap \mathcal{H}$, and so every G_δ-set in $2^{\beta X}$ containing F meets 2^X. Therefore $F \in Q$, and so $Q_1 \subseteq Q$. We complete the proof by showing that Q_1 is G_δ-closed in $2^{\beta X}$. Let $F \in 2^{\beta X} - Q_1$. Then there is a zero-set Z in βX such that $F \subseteq Z$ but $F \nsubseteq \mathrm{cl}_{\beta X}(Z \cap X)$. Let $\mathcal{H} = B(Z; \beta X - \mathrm{cl}_{\beta X}(Z \cap X))$. Then clearly $\mathcal{H}$ is a G_δ in $2^{\beta X}$ and $F \in \mathcal{H}$. Obviously $\mathcal{H} \cap Q_1 = \emptyset$. Thus Q_1 is G_δ-closed in $2^{\beta X}$. □

Theorem 2.4. *Let X be a normal, Hausdorff space. The following are equivalent:*

(i) 2^X *is G_δ-closed in* $2^{\beta X}$;

(ii) X *is Lindelöf.*

Proof. (i) ⇒ (ii). Assume that 2^X is G_δ-closed in $2^{\beta X}$. In the notation of the preceding lemma, this means that $2^X = Q_1$. We claim that X is Lindelöf. For the sake of contradiction, assume X is not Lindelöf. Then, there is a family $\mathcal{D}$ of closed subsets of X with the countable intersection property such that $\bigcap \mathcal{D} = \emptyset$. Let $\mathcal{D}_1$ be the family of countable intersections of members of $\mathcal{D}$. Then $\mathcal{D}_1$ is closed under countable intersection, and $\bigcap_{A\in\mathcal{D}_1} A = \emptyset$. Let $R = \bigcap_{A\in\mathcal{D}_1} \mathrm{cl}_{\beta X} A$. Then, since $R \cap X = \emptyset$, we have $R \in 2^{\beta X} - 2^X$. Let Z be any zero-set in βX containing R. Write $Z = \bigcap_{n\in\mathrm{N}} G_n$, where each G_n is open in βX. Now, for each n, $\bigcap_{A\in\mathcal{D}_1} \mathrm{cl}_{\beta X} A \subseteq G_n$, and so, by compactness, there is, for each n, a finite subset $\mathcal{F}_n$ of $\mathcal{D}_1$ so that $\bigcap_{A\in\mathcal{F}_n} \mathrm{cl}_{\beta X} A \subseteq G_n$. Let $\mathcal{F} = \bigcup_{n\in\mathrm{N}} \mathcal{F}_n$. Then $\mathcal{F}$ is a countable subset of $\mathcal{D}_1$ and $\bigcap_{A\in\mathcal{F}} \mathrm{cl}_{\beta X} A \subseteq \bigcap_{n\in\mathrm{N}} G_n = Z$. But $\bigcap_{A\in\mathcal{F}} A \in \mathcal{D}_1$ and so

$$R \subseteq \mathrm{cl}_{\beta X}(\bigcap_{A\in\mathcal{F}} A) \subseteq \bigcap_{A\in\mathcal{F}} \mathrm{cl}_{\beta X} A \subseteq Z\,.$$

It follows easily that $R \subseteq \mathrm{cl}_{\beta X}(Z \cap X)$. We have thus shown that. for $Z \in Z(\beta X)$,

$$R \subseteq Z \Rightarrow R \subseteq \mathrm{cl}_{\beta X}(Z \cap X)\,.$$

This means that $R \in Q_1$. But this is nonsense, since $R \notin 2^X = Q_1$. This shows that (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i). Assume X is Lindelöf. We show that $2^X = Q_1$, for which it suffices to show $Q_1 \subseteq 2^X$. So let $F \in Q_1$. Then

$$F = \bigcap\{\mathrm{cl}_{\beta X}(Z \cap X) \colon Z \in Z(\beta X), F \subseteq Z\}.$$

Let $\mathcal{K} = \{Z \in Z(\beta X) \colon F \subseteq Z\}$. We claim that $F = \mathrm{cl}_{\beta X}[\bigcap_{Z\in\mathcal{K}}(Z \cap X)]$. If possible, let $p \in F - \mathrm{cl}_{\beta X}[\bigcap_{Z\in\mathcal{K}}(Z \cap X)]$. Find a closed neighborhood N of p in βX such that $N \cap [\bigcap_{Z\in\mathcal{K}}(Z \cap X)] = \emptyset$. Since X is Lindelöf, there is a sequence $Z_1, Z_2, \ldots$ from $\mathcal{K}$ such that $N \cap [\bigcap_{i\in\mathrm{N}}(Z_i \cap X)] = \emptyset$. But $\bigcap_{i\in\mathrm{N}} Z_i \in \mathcal{K}$, and so $F \subseteq \mathrm{cl}_{\beta X}[\bigcap_{i\in\mathrm{N}}(Z_i \cap X)]$. But this implies $p \in \mathrm{cl}_{\beta X}[\bigcap_{i\in\mathrm{N}}(Z_i \cap X)]$, and so $N \cap [\bigcap_{i\in\mathrm{N}}(Z_i \cap X)] \neq \emptyset$. This contradiction proves that $F \subseteq \mathrm{cl}_{\beta X}[\bigcap_{Z\in\mathcal{K}}(Z \cap X)]$. Since the reverse inclusion holds trivially, we conclude that $F = \mathrm{cl}_{\beta X}[\bigcap_{Z\in\mathcal{K}}(Z \cap X)]$. Thus $F \in 2^X$, and so $Q_1 \subseteq 2^X$. Therefore 2^X is G_δ-closed in $2^{\beta X}$. □

Remark 2.5. It should be observed that the results of 2.3 and 2.4 carry over to higher cardinals. Calling a set a G_m-set if it is the intersection of m open sets (m denotes an infinite cardinal) and recalling that a space is m-Lindelöf if each of its open covers has a subcover of $\leqslant m$ sets, we see that, with obvious modifications, 2.3 and 2.4 hold with G_δ replaced by G_m, and Lindelöf replaced by m-Lindelöf.

3. Some remarks on the realcompactness of 2^X

By 2.4 and 2.1 it follows that if X is Lindelöf, then 2^X is realcompact. Indeed, if X is Lindelöf, then $2^{\beta X}$ is a compactification of 2^X in which 2^X is G_δ-closed. We now give a direct proof of this result. Recall that a completely regular space Y is realcompact if and only if every Z-ultrafilter on Y with the countable intersection property is fixed (see [1]).

Theorem 3.1. *Let X be Lindelöf and completely regular. Then 2^X is realcompact.*

Proof. Since a Lindelöf, completely regular space is normal, we conclude by [6, 4.9] that 2^X is completely regular (and Hausdorff) when X is completely regular and Lindelöf. We use the above characterization of realcompactness. So let θ be a Z-ultrafilter on 2^X with the countable intersection property, with X assumed to be Lindelöf. We define two families of sets as follows. We set

$$\alpha = \{A \in 2^X\colon \text{there exists } \mathcal{B} \in \theta \text{ such that } \mathcal{B} \subseteq 2^A\}\ ,$$

$$\beta = \{A \in 2^X\colon \text{there exists } \mathcal{B} \in \theta \text{ such that } \mathcal{B} \subseteq B(X;A)\}.$$

For $A \in \beta$, we define $\mathcal{G}_A = \{F \cap A\colon F \in \alpha\}$. We claim that for each $A \in \beta$, $\mathcal{G}_A$ has the countable intersection property. Let $\{F_1, F_2, ...\} \subseteq \alpha$, and let $A \in \beta$. Then, for each n, there is a set $\mathcal{B}_n \in \theta$ with $\mathcal{B}_n \subseteq 2^{F_n}$, and there is a set $\mathcal{B} \in \theta$ with $\mathcal{B} \subseteq B(X;A)$. Since θ has the countable intersection property, we have

$$\emptyset \neq (\bigcap_{n\in\mathrm{N}} \mathcal{B}_n) \cap \mathcal{B} \subseteq (\bigcap_{n\in\mathrm{N}} 2^{F_n}) \cap B(X;A)\ .$$

Any element in the latter intersection is contained in $\bigcap_{n\in\mathrm{N}} F_n$ and meets A. So, in particular, $(\bigcap_{n\in\mathrm{N}} F_n) \cap A \neq \emptyset$. Thus, each $\mathcal{G}_A$ has the countable intersection property. Since X is Lindelöf, there is, for each $A \in \beta$, a point $p_A \in \bigcap \mathcal{G}_A$. Let $L = \mathrm{cl}_X\,\{p_A\colon A \in \beta\}$. We now show that $L \in \bigcap \theta$, whence θ is fixed, and so 2^X is realcompact. We assume $L \notin \bigcap \theta$ and we will derive a contradiction. If $L \notin \bigcap \theta$, then there is a set $\mathcal{B} \in \theta$ such that $L \notin \mathcal{B}$. Now, since X is normal, it is easy to see that the sets of the form $B(X;Z_0) \cup 2^{Z_1} \cup ... \cup 2^{Z_n}$, where $Z_0, Z_1, ..., Z_n$ are zero-sets in X, form a base for the closed sets in 2^X. Now $\mathcal{B} \in \theta$ and so $\mathcal{B}$ is a zero-set in 2^X, and is, in particular, closed. Since $L \notin \mathcal{B}$, we can find zero-sets $Z_0, Z_1, ..., Z_n$ in X such that $\mathcal{B} \subseteq B(X;Z_0) \cup 2^{Z_1} \cup ... \cup 2^{Z_n}$, and $L \notin B(X;Z_0) \cup 2^{Z_1} \cup ... \cup 2^{Z_n}$. Now, if Z is a zero-set in X, then 2^Z is a zero-set in 2^X (see [2]).

We cannot have any $2^{Z_i} \in \theta$, because this would put Z_i in α and would imply $L \subseteq Z_i$, or equivalently $L \in 2^{Z_i}$, by the construction of L. So, since θ is a Z-ultrafilter, there is, for each $i = 1, 2, ..., n$, a zero-set $\mathcal{B}_i$ in θ such that $\mathcal{B}_i \cap 2^{Z_i} = \emptyset$. Letting $\mathcal{C} = \mathcal{B} \cap \bigcap_{i=1}^n \mathcal{B}_i$, we have $\mathcal{C} \in \theta$ and $\mathcal{C} \subseteq \mathcal{B}(X; Z_0)$. This implies $Z_0 \in \beta$, and so $p_{Z_0} \in Z_0 \cap L$. But $L \notin B(X; Z_0)$, so that $L \cap Z_0 = \emptyset$. This is a contradiction. We conclude that $L \in \bigcap \theta$. □

Remark 3.2. In [7], the realcompactness of 2^X is approached by uniformities, and 3.1 can be deduced as a corollary of results proved therein.

It does not seem to be known whether 2^X is realcompact whenever X is. Of course, if 2^X is realcompact, then X is, since (for Hausdorff X) the singletons in 2^X form a closed subspace homeomorphic to X. We have to be slightly careful in discussing the realcompactness of 2^X; since 2^X is completely regular only when X is normal. If we use the definition of realcompactness in [5], which applies in the non-completely regular setting, we can then meaningfully ask whether 2^X is realcompact when X is completely regular and realcompact.

Proposition 3.3. *Let $\mathcal{P}$ be a closed hereditary topological property. Let X be a regular, Hausdorff space such that $2^X \in \mathcal{P}$. If Y is a continuous-open-closed image of X, then $2^Y \in \mathcal{P}$.*

Proof. Let X be regular and Hausdorff, with $2^X \in \mathcal{P}$. Let $f : X \to Y$ be a continuous, open, and closed surjection. Define $F : 2^X \to 2^Y$ by $F(A) = f(A)$, and define $G : 2^Y \to 2^X$ by $G(B) = f^{-1}(B)$. By [6, 5.10.1, 5.10.2], F and G are continuous. Let $\mathcal{Y} = G(2^Y)$. Then $F \mid \mathcal{Y}$ and G are mutually inverse homeomorphisms between $\mathcal{Y}$ and 2^Y, and $G \circ F$ is a retraction of 2^X onto $\mathcal{Y}$. Now, since X is regular and Hausdorff, 2^X is Hausdorff (see [6, 4.9]). As a retract of 2^X, $\mathcal{Y}$ is therefore closed in 2^X. Thus 2^Y is homeomorphic to a closed subspace of 2^X. Since $\mathcal{P}$ is closed hereditary, $2^Y \in \mathcal{P}$. □

In particular, 3.3 is valid for the property realcompactness, so we can deduce the following. Let X be completely regular. If 2^X is realcompact, then X is realcompact and every continuous-open-closed completely regular image of X is realcompact. It does not seem to be known whether realcompactness is preserved under continuous-open-closed images. A counterexample would provide an example of a realcompact space whose hyperspace is not realcompact.

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